



TITLE:

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Theory and its related Fields)

AUTHOR(S):

Hirata, Kentaro

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CITATION:

Hirata, Kentaro. Limits at infinity of superharmonic functions and solutions of semilinear elliptic equations of Matukuma type (Potential Theory and its related Fields). 数理解析研究所講究録 2009, 1669: 52-57

ISSUE DATE:

2009-11

URL:

<http://hdl.handle.net/2433/141133>

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# Limits at infinity of superharmonic functions and solutions of semilinear elliptic equations of Matukuma type

Kentaro Hirata\*

Akita University

## 1 Motivation and Question

In 1930, Matukuma introduced the following semilinear elliptic equation

$$-\Delta u(x) = \frac{u(x)^p}{1 + \|x\|^2} \quad \text{in } \mathbb{R}^3$$

to study a gravitational potential  $u$  of a globular cluster of stars. Here  $\Delta$  is the Laplacian,  $\|x\|$  the Euclidean norm of a point  $x$ , and  $p > 1$  is a constant. This equation was deduced from Poisson's equation under several hypotheses in astrophysics. For details, see [8].

For the last several decades, many mathematicians have studied the existence of positive solutions of semilinear elliptic equations of the form

$$-\Delta u(x) = V(x)u(x)^p \quad \text{in } \Omega, \tag{1.1}$$

where  $V$  is a measurable function on a domain  $\Omega$  in  $\mathbb{R}^n$  with appropriate properties and the equation is understood in the sense of distributions. There are a great number of papers, but we mention only results relating to this talk.

- Kenig and Ni [4] studied (1.1) in the case of  $\Omega = \mathbb{R}^n$  ( $n \geq 3$ ). Indeed, they proved that if  $V$  is a measurable function on  $\mathbb{R}^n$  such that

$$|V(x)| \leq \frac{A}{(1 + \|x\|^2)^{1+\varepsilon}}$$

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\*e-mail: hirata@math.akita-u.ac.jp

for some  $\varepsilon > 0$  and  $A > 0$ , then (1.1) has bounded positive solutions.

- Zhao [7] generalized their result as follows. Let  $\Omega$  be an unbounded domain in  $\mathbb{R}^n$  ( $n \geq 3$ ) with a compact Lipschitz boundary or  $\Omega = \mathbb{R}^n$ . If  $V$  is a Green-tight function on  $\Omega$  and  $\alpha > 0$  is sufficiently small, then there are positive solutions  $u$  of (1.1) satisfying

$$\lim_{x \rightarrow \infty} u(x) = \alpha.$$

- The corresponding result for two dimensions was obtained by Ufuktepe and Zhao [6]. Let  $\Omega$  be an unbounded domain in  $\mathbb{R}^2$  with a compact boundary consisting of finitely many Jordan curves. If  $V$  is a Green-tight function on  $\Omega$  and  $\alpha > 0$  is sufficiently small, then there are positive solutions  $u$  of (1.1) satisfying

$$\lim_{x \rightarrow \infty} \frac{u(x)}{\log \|x\|} = \alpha.$$

In view of the last two results, the following question arises naturally.

**Question.** *Let  $\Omega$  be an unbounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ) with a compact boundary or  $\Omega = \mathbb{R}^n$  and let  $V$  be a nonnegative measurable function on  $\Omega$  with appropriate properties. Does every positive solution  $u$  of (1.1) satisfy*

$$\lim_{x \rightarrow \infty} u(x) = \alpha \quad (n \geq 3)$$

or

$$\lim_{x \rightarrow \infty} \frac{u(x)}{\log \|x\|} = \alpha \quad (n = 2)$$

for some  $\alpha \geq 0$ ?

**Remark 1.1.** When  $n \geq 3$  and  $V$  is a negative function with suitable properties, there is a positive solution  $u$  of  $-\Delta u = Vu^p$  in  $\mathbb{R}^n$  such that  $u(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ . See [1, 5] (ODE or PDE) and [2] (potential theoretic proof), etc. Thus the above question is significant in the case that  $V$  is nonnegative.

## 2 Notation and Convention

In the rest of this note, we let  $n \geq 2$  and suppose that  $\Omega$  is an **unbounded domain in  $\mathbb{R}^n$  with a compact boundary or  $\Omega = \mathbb{R}^n$** . The symbol  $A$  stands for an absolute positive constant whose value is unimportant and may change from line to line. Denote by  $B(x, r)$  the open ball of center  $x$  and radius  $r$ . A function  $u : \Omega \rightarrow (-\infty, +\infty]$  is called *superharmonic* if

- (i)  $u \not\equiv +\infty$ ,
- (ii)  $u$  is lower semicontinuous on  $\Omega$ ,
- (iii)  $u(x) \geq \frac{1}{\nu_n r^n} \int_{B(x, r)} u(y) dy$  whenever  $\overline{B(x, r)} \subset \Omega$ .

Here  $\nu_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . It is well known that for a superharmonic function  $u$  on  $\Omega$ , there is a unique nonnegative measure  $\mu_u$  such that

$$\int_{\Omega} \phi(x) d\mu_u(x) = \int_{\Omega} u(x)(-\Delta\phi(x)) dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

We discuss superharmonic functions  $u$  such that  $\mu_u$  is **absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}^n$** . Then the Radon-Nykodým derivative is denoted by  $f_u$ . It is obvious that  $f_u = -\Delta u$  for  $u \in C^2(\Omega)$ .

## 3 Main Results

This section presents our main results (answers to the question in Section 1).

**Theorem 3.1.** *Let  $n \geq 3$ . Suppose that*

$$0 \leq p < \frac{n}{n-2}.$$

*If  $u$  is a positive superharmonic function on  $\Omega$  satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2} u(x)^p \quad \text{for a.e. } x \in \Omega \setminus B(0, R)$$

*with some  $c > 0$  and  $R > 0$ , then  $u$  has a finite limit at infinity.*

As seen in the following, the bound  $p < n/(n - 2)$  is nearly optimal in Theorem 3.1. The case  $p = n/(n - 2)$  is still unsolved.

**Theorem 3.2.** *Let  $n \geq 3$  and  $c > 0$ . If*

$$p > \frac{n}{n - 2},$$

*then for each  $\beta > 0$ , there is a positive function  $u \in C^2(\mathbb{R}^n)$  such that*

$$0 \leq -\Delta u(x) \leq \frac{c}{1 + \|x\|^2} u(x)^p \quad \text{in } \mathbb{R}^n$$

*and*

$$\limsup_{x \rightarrow \infty} \frac{u(x)}{\|x\|^\beta} = +\infty.$$

Two dimensional result corresponding to Theorem 3.1 is as follows.

**Theorem 3.3.** *Let  $n = 2$  and let  $p \geq 0$  be arbitrary constant. If  $u$  is a positive superharmonic function on  $\Omega$  satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2 (\log \|x\|)^p} u(x)^p \quad \text{for a.e. } x \in \Omega \setminus B(0, R) \quad (3.1)$$

*with some  $c > 0$  and  $R > 1$ , then  $u(x)/\log \|x\|$  has a finite limit at infinity.*

## 4 Outline of proofs of Theorems 3.1 and 3.3

In this section, we give a sketch of the proof of Theorem 3.1 as well as Theorem 3.3. For details, see [3].

**Lemma 4.1.** *Let  $\{z_i\}$  be a sequence in  $\Omega$  with  $z_i \rightarrow \infty$  ( $i \rightarrow +\infty$ ). If  $v$  is a positive superharmonic function on  $\Omega$  such that*

$$f_v(x) \leq \frac{A}{\|x\|^2} \quad \text{for a.e. } x \in \bigcup_i B(z_i, \rho \|z_i\|)$$

*with some  $A > 0$  and  $0 < \rho \leq 1/2$ , then the following hold:*

- (i) *if  $n \geq 3$ , then  $v(z_i)$  has a finite limit as  $i \rightarrow \infty$ ;*
- (ii) *if  $n = 2$ , then  $v(z_i)/\log \|z_i\|$  has a finite limit as  $i \rightarrow \infty$ .*

Here the value of the limit is independent of  $\{z_i\}$ .

Indeed, this lemma is a special case  $p = 0$  of Theorems 3.1 and 3.3. When  $p > 0$ , the following lemma plays an essential role.

**Lemma 4.2.** *Let  $n \geq 3$ . Suppose that*

$$0 < p < \frac{n}{n-2}.$$

*Let  $u$  be a positive superharmonic function on  $\Omega$  satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2} u(x)^p \quad \text{for a.e. } x \in \Omega \setminus B(0, R)$$

*with some  $c > 0$  and  $R > 0$ . If  $\{z_i\}$  is a sequence in  $\Omega$  with  $z_i \rightarrow \infty$  ( $i \rightarrow +\infty$ ), then there exist  $A > 0$  and  $i_0, \ell \in \mathbb{N}$  such that*

$$u \leq A \quad \text{on } \bigcup_{i \geq i_0} B(z_i, 2^{-\ell-3}\|z_i\|).$$

The proof is based on arguments of minimal fine topology and nonlinear analysis. The corresponding result for two dimensions is as follows.

**Lemma 4.3.** *Let  $n = 2$  and let  $p > 0$  be arbitrary constant. Let  $u$  be a positive superharmonic function on  $\Omega$  satisfying*

$$f_u(x) \leq \frac{c}{\|x\|^2 (\log \|x\|)^{p-1}} u(x)^p \quad \text{for a.e. } x \in \Omega \setminus B(0, R)$$

*with some  $c > 0$  and  $R > 1$ . If  $\{z_i\}$  is a sequence in  $\Omega$  with  $z_i \rightarrow \infty$  ( $i \rightarrow +\infty$ ), then there exist  $A > 0$  and  $i_0 \in \mathbb{N}$  such that*

$$\frac{u(x)}{\log \|x\|} \leq A \quad \text{for } x \in \bigcup_{i \geq i_0} B(z_i, 2^{-5}\|z_i\|).$$

Now, Theorem 3.1 is proved immediately. Let  $\{z_i\}$  be arbitrary sequence in  $\Omega$  with  $z_i \rightarrow \infty$  ( $i \rightarrow +\infty$ ). By Lemma 4.2,

$$\begin{aligned} f_u(x) &\leq \frac{c}{\|x\|^2} u(x)^p \\ &\leq \frac{A}{\|x\|^2} \quad \text{for a.e. } x \in \bigcup_{i \geq i_0} B(z_i, 2^{-\ell-3}\|z_i\|). \end{aligned}$$

By Lemma 4.1,  $u(z_i)$  has a finite limit and its value is independent of  $\{z_i\}$ . Thus Theorem 3.1 follows.

The proof of Theorem 3.3 is similar.

## 5 Conjecture

In the proof of Lemma 4.2, we assumed  $p < \frac{n}{n-2}$  to use the fact

$$\|\cdot\|^{2-n} \in L_{loc}^q \quad \text{for some } q > p.$$

I do not have other techniques, but we expect that

Theorem 3.1 holds for  $p = \frac{n}{n-2}$  as well.

i.e.

$$f_u(x) \leq \frac{C}{\|x\|^2} u(x)^{\frac{n}{n-2}} \implies \lim_{x \rightarrow \infty} u(x) \text{ exists.}$$

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